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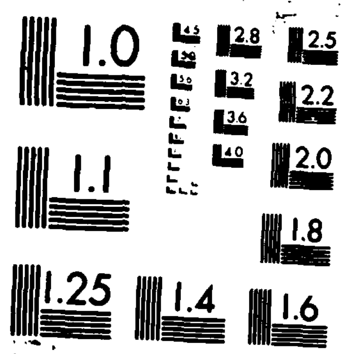

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Research Report CCS 574

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by

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ABSTRACT

The portfolio selection problem with one safe asset and n risky assets is analyzed via a new decision theoretic criterion: the Optimized Certainty Equivalent (OCE), recently introduced by the authors. Fundamental results in portfolio theory, previously studied under the Expected Utility criterion (EU), such as separation Theorems, comparative statistics analysis, and threshold values for inclusion or exclusion of risky assets in the optimal portfolio, are obtained here. In contrast to the EU model, our results for the OCE maximizing investor do not impose restrictions on either the utility function or the underlying probability laws. We also derive a dual portfolio selection problem and provide it with a concrete economic interpretation.

Key Words

Portfolio Selection, Expected Utility, Risk Aversion, Optimized Certainty Equivalent

1. Introduction

The analytical study of portfolio selection problems under uncertainty was initiated by the work of Markovitz [10] and Tobin [18] in the framework of mean variance analysis, in which they obtained (for the single time period case) a number of important results including the first separation theorems. The simplicity of their results attracted a great deal of attention and much effort has been directed toward their generalization within the von-Neumann Morgenstern expected utility criterion, see e.g. Arrow [1], Sharpe [17], Sandmo [15], Cass and Stiglitz [6], and many others. [1]

For an expected utility maximizing risk averse investor the classical portfolio selection problem over a fixed period of time consists of allocating the fraction x_i of his initial wealth w to the ith risky asset ($i = 1, \dots, n$) whose (random) return per dollar is R_i, and a fraction x_0 to a riskless asset, whose fixed rate of return is ρ. The investor's wealth at the end of the period is $w\rho x_0 + w \sum_{i=1}^{n} R_i x_i$. An *optimal portfolio* is then the solution of the following problem:

$$\max \{Eu(w\rho x_0 + w \sum_{i=1}^{n} R_i x_i) : x_0 + \sum_{i=1}^{n} x_i = 1 , x_i \geq 0 , x_0 \geq 0\} \tag{1.1}$$

where u is the investor's concave utility function, defined on the final wealth, and E denotes mathematical expectation with respect to the random vector $R := (R_1, \dots R_n)^T$. The concavity

1. There have been, of course, other approaches of treating portfolio problems and portfolio dynamics. In particular we mention the Chance Constrained programming approach of Charnes and Cooper [7]. One special case of implementing this approach is to maximize the probability of achieving a final wealth which is at least as large as some "goal" value. See also Charnes and Thore [8].

assumption on the utility function u is equivalent to the supposition that the investor is risk averse, i.e. for any random variable Z, he prefers the sure payment E(Z) to the random payoff Z.

When the utility function u is strictly increasing, (with inverse u^{-1}) an equivalent problem to (1.1) is obtained by replacing the objective function by $u^{-1}Eu\,(\rho w x_0 + w\Sigma R_i x_i)$ i.e. the investor maximizes the certainty equivalent of the final wealth. Recall that for a random variable Z,

$$C(Z) := u^{-1}Eu(Z)$$

is the *Certainty Equivalent* (CE) of Z; it is the sure payment for which the decision maker remains indifferent to the lottery Z i.e. $u(C(Z)) = Eu(Z)$.

In this paper we use a new decision theoretic criterion in order to analyze the portfolio selection problem. According to this decision criterion, a random variable X is preferred to a random variable Y if

$$S_u(X) \geq S_u(Y)$$

where $S_u(Z)$ is the *Optimized Certainty Equivalent* (OCE) of the random variable Z:

$$S_u(Z) := \sup_{\eta \in R} \{\eta + Eu(Z-\eta)\}$$

Here u is a normalized strictly increasing and strictly concave utility function (see Section 2). The OCE was first introduced in Ben-Tal and Teboulle [4], [5]. Further properties and applications in simple microeconomic models under uncertainty are discussed in Ben-Tal and Ben-Israel [3]. The main properties of the OCE needed for our purposes here are summarized in Section 2.

The OCE maximizer investor determines the optimal investment fractions $(x_0^*, x_1^*, \ldots, x_n^*)$ by solving

$$\max S_u\,(\rho w x_0 + w \sum_{i=1}^{n} x_i R_i)$$

subject to

$$x_0 + \sum_{i=1}^{n} x_i = 1$$

$$x_0 \geq 0,\ x_i \geq 0,\ i = 1, \ldots, n.$$

We denote by $y_i = wx_i$ the *amount* invested in the ith asset. Then $\xi = y_1 + \ldots y_n$ is called the *portfolio size* and $(y_1/\xi, \ldots, y_n/\xi)$ is the *composition* of the portfolio.

The results we obtain for the portfolio problem extends previous results in the sense that restrictions on the utility function u (such as constant absolute risk aversion, or other "third derivative assumption") are lifted. This is due partly to the *shift additivity* of the OCE

$$S_u(Z + \text{constant}) = S_u(Z) + \text{constant}$$

We also point out that the OCE coincides with the classical CE for a constant absolute risk aversion utility function.

In section 4 we derive "separation theorems" expressing the invariance of the optimal portfolio $y_1^*, \ldots y_n^*$ with respect to the initial wealth for any diversifier $(y_0^* > 0)$ investor.

Another type of separation result contained in this section is preceded by the discussion in Section 3. There we show that any given utility u, generates a class of utilities M_u, which contains the entire spectrum of degrees of risk aversion, ranging from a *maximal risk averter* (one whose certainty equivalent is the smallest value of the random variable) to a *risk neutral* investor (whose certainty equivalent is the mean). A utility $v \in M_u$ is characterized by a single parameter α which we call the *risk index* of v. We then prove in section 4 that, for any u, all investors with utilities in M_u have the same composition of risky assets in the optimal portfolio.

Separation theorems of the above kind for the EU-model were obtained previously either by restricting u to a particular class (e.g. Cass and Stiglitz [6],) or by restricting the random vector $(R_1, \ldots, R_n)$ to a particular class of probability distributions (e.g. Ross [14]).

Certain results require both types of restriction. A well known example is the *hybrid model* assuming exponential utility and a normal distribution (e.g. Bamberg and Spremann [2]).

Another important question in the theory of portfolio selection is the nature of the effect of a change in the safe rate of return ρ on the optimal portfolio size ξ^*. This problem is addressed in Section 5, in which it is shown that for a complete diversifier risk averse investor ($0 < y_0^* < w$) ξ^* decreases as ρ increases. For an expected utility maximizer, with only one risky asset, a similar result was obtained by Fishburn and Porter [9], but only under additional restrictions, requiring third derivative assumptions.

The case of independently distributed risky assets is studied in section 6. For this case it is possible to decide whether the ith risky asset should or should not be included in the optimal portfolio by simply comparing its mean $E(R_i)$ to the safe return ρ. Corresponding results for the EU model are included in the paper by McEntire [11], who extends earlier results of Samuelson [16]. Section 6 is concluded by deriving an *explicit formula* for the optimal portfolio in the case of one risky asset ($n=1$), having only two possible outcomes.

Section 7 deals with the case of quadratic utility. We obtain a necessary and sufficient condition for complete diversification, and for the case of independent $\{R_i\}$, an explicit formula for the optimal investment amounts $y_1^*, \ldots, y_n^*$.

In a recent paper [4] the authors developed a comprehensive duality theory for a pair of stochastic programs, the primal having stochastic constraints and the dual having a stochastic objective function. In this duality framework, the stochasticity in the objective function is treated by maximizing its OCE. This duality framework is specialized in Section 8 to produce a *dual Portfolio Selection problem*, which is further provided with a concrete interpretation. To the best of our knowledge, such duality results do not exist in the literature of the Portfolio Selection problem under the Expected Utility approach.

2. The Optimized Certainty Equivalent: Basic Properties.

Let U_N denote the class of strictly increasing, strictly concave continuously differentiable and normalized utility functions, i.e.,

$$U_N = \{ u \in C^1 : u' > 0 , u \text{ strictly concave}, u(0) = 0 , u'(0) = 1 \}$$

Given $u \in U_N$, the Optimized Certainty Equivalent (OCE) of a random variable X is defined by

$$S_u(X) = \sup_{\eta \in \mathbb{R}} \{ \eta + Eu(X-\eta) \} \tag{2.1}$$

The OCE was first introduced in Ben-Tal and Teboulle [4]. Further properties and applications in economic models of decisions under uncertainty are discussed in Ben-Tal and Ben-Israel [3]. We collect below properties of the OCE needed in this paper, see [4] for proofs.

Proposition 2.1 For any utility function $u \in U_N$ and a constant c

(a) *Constancy* $S_u(c) = c$

(b) *Risk Aversion* $S_u(X) < E(X)$ for all non degenerate random variables X

(c) *Lower Bound* If X is a random variable and $X \geq x_{min}$, then

$S_u(X) \geq x_{min}$

(d) *Shift Additivity* $S_u(X+c) = S_u(X) + c$

As an example of the OCE, consider the *exponential utility* function defined by $u(t) = 1 - e^{-t}$. By simple calculus, it follows from (2.1) that

$$S_u(X) = -\log Ee^{-X} \tag{2.2}$$

On the other hand since $u \in U_N$, u^{-1} exists and the classical certainty equivalent CE is here

$$C(X) = u^{-1}Eu(X) = -\log Ee^{-X} = S_u(X)$$

showing that for the exponential utility the certainty equivalent CE coincides with the OCE.

In the rest of the paper we use the subclass of utilities:

$$U := \{ u \in U_N : u \text{ twice differentiable}, u'' < 0 \}.$$

Proposition 2.2 Let Z be a nondegenerate random vector in $\mathbb{R}^n$ and let $u \in U$. Define the *OCE functional of Z* by

$$s(y) := S_u(y^TZ) = \sup_{\eta \in \mathbb{R}} \{\eta + Eu(y^TZ - \eta)\} \qquad , y \in \mathbb{R}^n \tag{2.3}$$

Then,

(a) The functional s is concave and it is given by

$$s(y) = \eta(y) + Eu(y^TZ - \eta(y)) \tag{2.4}$$

where $\eta(y)$ is the unique solution of the equation

$$Eu'(y^TZ - \eta) = 1 \text{ , for all } y \in \mathbb{R}^n \tag{2.5}$$

(b) $\eta(y)$ and $s(y)$ are continuously differentiable functions satisfying $\eta(0) = s(0) = 0$ and $\nabla\eta(0) = \nabla s(0) = E(Z)$.

(c) The Hessian matrix $\nabla^2 s(\cdot)$ is negative definite and in particular $\nabla^2 s(0) = u''(0)V_Z$ where V_Z is the covariance matrix of the random vector Z.

Proof

(a) and (b) are proven in [4], Proposition 6. We proceed to the proof of (c). By differentiating (2.5) with respect to y we obtain:

$$\nabla\eta(y) = Eu''(y^TZ-\eta)Z \,/\, Eu''(y^TZ-\eta) \tag{2.6}$$

Differentiating twice (2.4) with respect to y and using (2.5) and (2.6) we get:

$$\nabla^2 s(y) = Eu''(y^TZ - \eta(y)\, ZZ^T - \frac{Eu''(y^TZ - \eta(y))\; ZEu''(y^TZ-\eta(y))Z^T}{Eu''(y^TZ - \eta(y))} \tag{2.7}$$

and thus $\nabla^2 s(0) = u''(0)\,\{E(ZZ^T) - E(Z)E(Z^T)\} = u''(0)V_Z$

Now for any $0 \neq x \in \mathbb{R}^n$ we compute

$$x^T\nabla^2 s(y)x = E\bar{u}''(Z^Tx)^2 - E^2\bar{u}''Z^Tx / E\bar{u}'' \tag{2.8}$$

where for simplicity of notation $\bar{u}''$ stands for $u''(y^TZ - \eta(y))$. Since u is strictly concave,

$\bar{u}''<0$ and $E\bar{u}'' < 0$. Applying Cauchy Schwartz inequality we have:

$$E^2\bar{u}''Z^Tx = E^2\sqrt{-\bar{u}''}\ Z^T x\cdot\sqrt{-\bar{u}''} < E\bar{u}''\ (Z^Tx)^2\ E\bar{u}'' \qquad (2.9)$$

with equality if Z^Tx = constant (i.e., if Z is degenerate). Hence, using (2.9) and the fact that $E\bar{u}'' < 0$, it follows from (2.8) that

$$x^T\ \nabla^2 s(y)x < 0 \qquad \forall 0 \neq x \in \mathbb{R}^n$$

Q.E.D

3. α - Modified Utilities and Measure of Risk Aversion

Consider the exponential utility $u(t) = 1 - e^{-t} \in U$ with the absolute Arrow-Pratt risk aversion measure [1], [12]:

$$r(t) := -u''(t) / u'(t) = 1 \tag{3.1}$$

For this utility a one parameter modification

$$u(t) \to 1/\alpha\ u(\alpha t),\ \alpha \geq 0\ \mathbf{2} \tag{3.2}$$

generates another exponential utility in U with $r(t) = \alpha$. As the parameter α increases from 0 to ∞ the degree of risk aversion increases. In the limit cases, $\alpha = 0$ corresponds to risk neutrality (The certainty equivalent $C(X) = E(X)$) while $\alpha = \infty$ corresponds to extreme risk aversion ($C(X) = x_{min}$), see Bamberg and Spreman [2] Theorems 1 and 2; the results are at any rate a special case of Theorem 3.1 below.

Consider now an arbitrary but fixed utility $u \in U$.

Definition A utility v is *α - modified* with respect to u if

$$v(t) = (1/\alpha)u(\alpha t) \quad \text{for some } \alpha > 0.$$

The set of all α - modified utilities is denoted by M_u i.e.,

$$M_u := \{u_\alpha : u_\alpha(t) = (1/\alpha)u(\alpha t),\ \alpha > 0\}$$

Clearly $M_u \subset U$. For a utility $v \in M_u$, the value of α such that $v(t) = (1/\alpha)u(\alpha t)$ is called the *risk index*, clearing the risk index of u itself is one. For $u_\alpha \in M_u$ the corresponding OCE is

$$S_\alpha(X) := S_{u_\alpha}(X) = \sup_{\eta \in iR} \{\eta + (1/\alpha) Eu(\alpha(X - \eta))\}$$

$$= (1/\alpha) \sup_{\xi} \{\xi + Eu(\alpha X - \xi)\} = (1/\alpha) S_u(\alpha X) \tag{3.3}$$

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2. Note that $\lim_{\alpha \to 0^+} u(\alpha t)/\alpha = t$ by l'Hopital's rule and the fact $u \in U$.

The next result shows that M_u contains the whole spectrum of the degrees of risk aversion for a wide class of utilities.

Theorem 3.1 Let $u \in U$, then,

(a) $\lim_{\alpha \to 0} S_\alpha(X) = E(X)$ [Risk neutrality]

(b) If u is *essentially smooth*[3], then for a random variable with infimum support $x_{min} > -\infty$

$$\lim_{\alpha \to \infty} S_\alpha(X) = x_{min}$$

(c) $S_\alpha(X)$ is monotone decreasing in α.

Proof

(a) and (b) follow from Lemmas 4.2 and 4.3 in Ben-Tal and Teboulle [5]. We proceed to prove (c). By (3.3)

$$S_\alpha(X) = (1/\alpha)\, S_u(\alpha X) = (s(\alpha)/\alpha)$$

where $s(\alpha)$ is defined in (2.3). By Proposition 2.2 $s(\alpha)$ is a concave differentiable function, hence

$$(d/d\alpha)(s(\alpha)/\alpha) = \alpha s'(\alpha) - s(\alpha))/\alpha^2 \leq 0 .$$

The inequality following from the gradient inequality for $s(\alpha)$ and the fact that $s(0) = 0$

Q.E.D.

Consider two decision makers A,B with corresponding utilities u_{α_A}, $u_{\alpha_B} \in M_u$. Following the result in Theorem 3.1 (c) we cay that *A is more risk averse than B* if the risk index of A is greater or equal the risk index of B, i.e., $\alpha_A \geq \alpha_B$.

The next result deals with the widely used (see e.g. [2], [6]) class of hyperbolic absolute risk aversion (HARA) utilities u, which are characterized by :

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3. See Rockafellar [13] p. 251. In particular the exponential, power and log utilities satisfy this assumption.

The next result deals with the widely used (see e.g. [2], [6]) class of hyperbolic absolute risk aversion (HARA) utilities u, which are characterized by:

$$r_u(t) = 1/(at + b) \qquad [b \geq 0 , at + b \geq 0] \qquad (3.4)$$

For such utilities we show that the ranking of risk aversion obtained by the risk index is the same as the ranking obtained in terms of the Arrow-Pratt measure.

Proposition 3.1 Let u be a HARA utility function. Then A is more risk averse than B if and only if

$$r_{u_{\alpha_A}}(t) \geq r_{u_{\alpha_B}}(t) \quad \text{For all } t.$$

Proof:

For any α - modified utility u_α note that

$$r_{u_\alpha}(t) = \alpha r_u(\alpha t) . \qquad (3.5)$$

Now, $r_{u_{\alpha_A}}(t) \geq r_{u_{\alpha_B}}(t)$ if and only if $\alpha_A r_u(\alpha_A t) \geq \alpha_B r_u(\alpha_B t)$

or equivalently, by using (3.4): $\alpha_A/(\alpha_A at + b) \geq \alpha_B/(\alpha_B at+b)$

which finally reduces to $\alpha_A \geq \alpha_B$

Q.E.D.

Many results in economic under uncertainty depend crucially on the assumption that r_u is a decreasing function (see Arrow [4]). The next result shows that this property is preserved under α - modification.

Proposition 3.2 A utility $u \in U$ has a decreasing absolute risk aversion measure $r_u(t)$ if and only if every $v \in M_u$ has this property.

Proof

Let $v \in M_u$, then $v(t) = (1/\alpha)u(\alpha t)$ for some $\alpha > 0$. Hence from (3.5)

$r_v(t) = \alpha r_u(\alpha t)$. Then $r_v(t)$ is decreasing if and only if

$$(d/dt)\, r_v(t) = (d/dt)(\alpha r_u(\alpha t) = \alpha^2 r_u{'}\ (\alpha t) \leq 0. ,$$

the latter inequality being valid if and only if $r_u(t)$ is decreasing.

Q.E.D.

Another commonly used postulate in economic theory is that the relative Arrow-Pratt measure $R_u(t) := t\, r_u(t)$ is an increasing function. It can be easily verified that this property remains valid for all utilities in M_u.

4. A Separation Theorem

Let x_0^* be the optimal fraction invested in the safe asset, and x_i^* $(i=1,\dots,n)$ the optimal fraction invested in the ith risky asset, for an investor maximizing the OCE of the final wealth, given that its initial wealth is w. So $x_0^* \in R$, $x^* \in R^n$ is the optimal solution of

$$(P) \quad \max_{x_0 \geq 0, x \geq 0} \{ S_u(wx_0\rho + wR^Tx) : x_0 + \sum_{i=1}^{n} x_i = 1 \}.$$

The main result of this section is the following separation theorem.

Theorem 4.1 Let $u \in U$ and assume a diversifier investor, i.e. $x_0^* > 0$. Then

(a) The optimal amounts $y_1^*, \dots, y_n^*$ invested in the risky assets are independent of the initial wealth w.

(b) The composition of the risky assets in the optimal portfolio is the same for all investors with utilities in M_u.

Proof:

(a) Let $y_i = wx_i$ be the amount invested in asset i, $i=0,1,\dots,n$. In terms of the y_i problem (P) is

$$\max_{\substack{y_0 \geq 0 \\ y \geq 0}} \{S_u(y_0\rho + R^Ty): y_0 + \sum_{i=1}^{n} y_i = w\}$$

Substituting $y_0 = w - \sum_{i=1}^{n} y_i$ in the objective function the problem reduces to

$$\max_{y \geq 0} \{S_u(\rho(w - \sum_{i=1}^{n} y_i) + R^Ty): \sum_{i=1}^{n} y_i \leq w\}$$

By the shift additivity of S_u (Proposition 2.1(d)) the objective function is

$$S_u(\rho(w-\Sigma y_i) + R^Ty) = \rho(w-\Sigma y_i) + S_u(R^Ty) = \rho w - \rho\Sigma y_i + s(y)$$

where s(.) is the OCE functional for the random vector R. so (P) can be rewritten as

$$\rho w + \max_{y \geq 0} \{-\rho \Sigma y_i + s(y): \Sigma y_i \leq w\} \tag{4.1}$$

Notice that the objective function is independent of w. Moreover since a diversifier investor is assumed, the constraint is inactive, i.e. $\Sigma y_i < w$. Hence the optimal solution of (4.1) $y^* = (y_1^*, \ldots, y_n^*)$ is independent of w.

(b) Consider a utility $v \in M_u$ i.e. $v(t) = (1/\alpha)u(\alpha t)$ for some $\alpha > 0$. By (3.3)

$$S_v(R^T y) = S_u(\alpha R^T y)/\alpha = s(\alpha y)/\alpha$$

Thus for an investor with utility v, problem (4.1) is

$$\max_{y \geq 0} \{-\rho \Sigma y_i + s(\alpha y)/\alpha : \Sigma y_i \leq w\} \tag{4.2}$$

and denote by $y^*(\alpha)$ its optimal solution. By Proposition 2.2 (a), $s(\alpha y)$ is concave hence the Karush-Kuhn-Tucker optimality conditions are necessary and sufficient, i.e. $y^*(\alpha)$ is optimal for problem (4.2) if and only if for all $j=1, \ldots, n$, there exists $\delta^* \geq 0$ such that:

$$(1/\alpha)(\partial/\partial y_j)s(\alpha y^*(\alpha)) - \rho \quad \begin{matrix} = \delta^* & \text{if } y_j^*(\alpha) > 0 \\ \leq \delta^* & \text{if } y_j^*(\alpha) = 0 \end{matrix} \tag{4.3}$$

$$\delta^*(\Sigma y^*(\alpha) - w) = 0$$

The assumption that the investor is a diversifier implies that $\Sigma y_i^*(\alpha) < w$, which in turn implies that $\delta^* = 0$ in (4.3).

Using Proposition 2.2 (a), we compute:

$$(1/\alpha)(\partial/\partial y_i)S(\alpha y^*(\alpha) = E\,(R_j u'(\alpha(R^T y^*(\alpha) - \eta(\alpha)))) \tag{4.4}$$

where $\eta(\alpha)$ is the unique solution of

$$Eu'(\alpha(R^T y^*(\alpha) - \eta(\alpha))) = 1 \tag{4.5}$$

Therefore the optimality conditions for $y(\alpha)$ are:

$$ER_j u'(\alpha(R^T y^*(\alpha) - n(\alpha)) - \rho \begin{cases} = 0 & \text{if } y_j(\alpha) > 0 \\ \\ \leq 0 & \text{if } y_j(\alpha) = 0 \end{cases} \tag{4.6}$$

Now let y^* be the optimal solution of problem (4.1) (i.e. for an investor with utility u that is $\alpha=1$). Thus y^*satisfies for all $j=1,\ldots,n$:

$$\partial/\partial y_j s(y^*) = ER_j u'(R^T y^* - \eta^*) - \rho \begin{cases} = 0 & \text{if } y_j^* > 0 \\ \leq 0 & \text{if } y_j^* = 0 \end{cases} \tag{4.7}$$

where η^* uniquely solves

$$Eu'(R^T y^* - \eta) = 1 \tag{4.8}$$

Hence, it is easily seen from (4.5), (4.6) that $y^*(\alpha) = y^*/\alpha$, $\eta^*(\alpha) = \eta^*/\alpha$ solve (4.7) (4.8). Therefore $y^*(\alpha) = y^*/\alpha$ is the optimal solution of (4.2). From this, it follows that

$$y_i^*(\alpha)/\Sigma y_j^*(\alpha) \quad = \quad y_i^*/\Sigma y_j^*$$

showing that the composition of the risky assets in the optimal portfolio is independent of α.

Remark 3.1 The proof (b) actually demonstrates that for an investor with utility $v \in M_u$, the optimal amount invested in each risky asset is inversely proportional to its degree of risk aversion α. Thus, as one should expect, a more risk averse investor will invest less money in the risky assets.

5. Effect of a Change in the Safe Return

The effect of a change in the safe return per dollar ρ on the optimal allocation of an investor maximizing the expected utility of its wealth was analyzed by Fishburn and Porter [9] for the special case of one risk free asset and one risky asset. The results obtained there required some additional conditions on the absolute risk aversion and relative risk aversion measures which by themselves make some of the conclusions controversial. In Ben-Tal and Ben-Israel [3] the same model was analyzed for an OCE maximizer investor; unambiguous results, not necessitating "third derivative assumption" on the utility function were obtained.

In this section we extend the results of [3] to the case of n risky assets. As in Section 4, let $y_1^*, \ldots, y_n^*$ be the optimal solution of

$$w\rho + \max_{y \geq 0} \{-\rho \Sigma y_i + s(y) : \Sigma y_i \leq w\} \tag{5.1}$$

If $y_i^* > 0$ for all $i = 0, \ldots, n$, the investor is said to be a *complete diversifier*. We denote by $\xi^* = y_1^* + \ldots + y_n^*$ the *optimal portfolio size*.

Theorem 5.1 For a complete diversifier investor, the optimal portfolio size ξ^* decreases as the rate of return for the safe return ρ increases.

Proof: For a complete diversifier, the optimality conditions (4.7) reduce to

$$\partial/\partial y_i s(y^*) = \rho \qquad i = 1, \ldots, n \tag{5.2}$$

By proposition 2.2(c) the Hessian matrix $\nabla^2 s(\cdot)$ is negative definite, thus non singular, and since here it coincides with the Jacobian of the system of equations (5.2), it follows from the implicit function theorem that there exists a solution $y^* = y(\rho)$ which is a continuously differentiable function. Therefore

$$\nabla s\,(y(\rho)) = \rho \cdot e \tag{5.3}$$

where $e := (1, \ldots, 1)^T$. Differentiating this identity with respect to ρ, we obtain

$$\nabla^2 s(y(\rho)) \cdot y'(\rho) = e \tag{5.4}$$

where $y'(\rho) := (y_1'(\rho), \ldots\ldots, y_n'(\rho))^T$.

Hence, $\nabla^2 s(\cdot)$ being non singular, it follows from (5.4) that:

$$y'(\rho) = [\nabla^2 s(y(\rho))]^{-1} e \tag{5.5}$$

Multiplying (5.5) by the row vector e^T we get

$$e^T y'(\rho) = \Sigma y_i'(\rho) = e^T[\nabla^2 s(y(\rho))]^{-1} e < 0$$

Therefore $d/d\rho(\xi^*) = d/d\rho \Sigma y_i(\rho) = \Sigma\, y_i'(\rho) < 0$. Q.E.D.

6. The Case of Independent Risky Assets

In this section we present two fundamental results for the portfolio problem with independent risky assets. The corresponding results for an Expected Utility maximizing investor are given in McEntire [11]. The case of one risky asset was treated previously in Fishburn and Porter [9].

We assume that the investor is a diversifier ($y_0^* > 0$) so the portfolio problem is as given in section 4:

$$\max_{\substack{y_i \geq 0 \\ \Sigma y_i < w}} \{-\rho \sum_{i=1}^{n} y_i + s(y)\} \tag{6.1}$$

Theorem 6.1 Assume that R_i is independent of the random variables $\{R_j\}j{\neq}i$. Then the i^{th} risky asset is not included in an optimal portfolio (i.e. $y_i^*=0$) if and only if $E(R_i) \leq \rho$.

Proof: <u>If</u>. From (4.7), (4.8) y^* is optimal for (6.1) if and only if for all $j=1, \ldots, n$

$$(\partial/\partial y_j)s(y^*) - \rho = E\, R_j u'(R^T y^* - \eta(y^*)) - \rho \begin{cases} = 0 & \text{if } y_j^* > 0 \\ \leq 0 & \text{if } y_j^* = 0 \end{cases} \tag{6.2}$$

where $\eta^* = \eta(y^*)$ is the unique solution of

$$Eu'(R^T y^* - \eta^*) = 1 \tag{6.3}$$

Now, since $y_i^* = 0$ is assumed, $\eta(y^*)$ depends on $\{y_j^*\}_{j \neq i}$ only and hence $u'(R^T y^* - \eta(y^*))$ depends on $\{y^*_j\}_{j \neq i}$ only. Therefore, by the independence of the random variable R_i from $\{R_j\}_{j \neq i}$, we have:

$$(\partial/\partial y_i)s(y^*) = E(R_i) \cdot Eu'(R^T y^* - \eta(y^*))$$

$$= E(R_i), \text{ for } y^* = (y^*_1, \ldots, y^*_{i-1}, 0, y^*_{i+1} \ldots y^*_n) \tag{6.4}$$

the last equality following from (6.3). Substituting (6.4) in (6.2) for $j=1$, we get

$$E(R_i) - \rho \leq 0$$

<u>only if</u> Let $E(R_i) \leq \rho$ and assume $y_i^* > 0$, then by (6.2)

$$0 = (\partial/\partial y_i)s(y^*) - \rho \leq (\partial/\partial y_i)s(y^*) - E(R_i). \tag{6.5}$$

The function $(\partial/\partial y_i)s(y^*_1, y^*_2, \ldots, y^*_{i-1}, \cdot, y^*_{i+1}, \ldots, y^*_n)$ is strictly decreasing by the strict concavity of $s(\cdot)$,hence

$$(\partial/\partial y_i)s(y^*_1, \ldots, y^*_{i-1}, \cdot, y^*_{i+1}, \ldots, y^*_n) < (\partial/\partial y_i)s(y^*_1, \ldots, y^*_{i-1}, 0, y^*_{i+1}, \ldots, y^*_n)$$

$$= E(R_i), \quad \text{by (6.4)}$$

This is a contradiction to (6.5), hence $y_i^* = 0$. Q.E.D.

We note that from Theorem 5.1 it follows that the safe return ρ is a *threshold value*; all risky assets whose mean return is less than or equal to ρ are excluded from the portfolio, while those with means larger than ρ are included in the optimal portfolio.

Our next result for independent assets is given in the following theorem which states that a risk averse investor will put all his wealth in the safe asset provided the safe return ρ is at least as large as the largest mean of the risky assets.

Theorem 5.1 Assume that $R_1, R_2, \ldots, R_n$ are independent random variables. Then the investor will put all his wealth w in the safe asset if and only if $\rho \geq E(R_i)$, all $i=1, \ldots, n$.

Proof: <u>Only if</u> Suppose that the investor put all his wealth w in the safe asset i.e. $y_0^* = w$. Then $y^* = (y_1^*, \ldots, y_n^*) = 0$ and $\delta^* = 0$ in the optimality conditions (4.3) (with $\alpha = 1$), which then reduces to $\partial/\partial y_i s(0) \leq \rho \quad i = 1, \ldots, n$.

But by Proposition 2.2 (b), $\partial s/\partial y_i s(0) = E(R_i)$, hence $\rho \geq E(R_i)$ for all $i = 1, \ldots, n$.

<u>If</u>. Let $\rho \geq E(Ri)$ for all $i = 1, \ldots, n$. If $\delta^* = 0$ in (4.3) then y_i^* is given by solving (6.2) - (6.3) and we proved in this case that $\rho \geq E(R_i)$ implies $y_i^* = 0$, hence

$y^* = (y_1^*, \ldots y_n^*) = 0$ and so $y_0^* = w$.

If $\delta^* > 0$ and $y_i^* > 0$ for some i, then from (6.3)

$$0 < \delta^* = (\partial/\partial y_i)s(y^*) - \rho < (\partial/\partial y_i)s(y^*_1, \ldots, y^*_{i-1}, 0, y^*_{i+1}, \ldots, y^*_n) - \rho$$
$$= E(R_i) - \rho$$

The above inequality follows from the strict concavity of s, which implies the monotonicity of $(\partial/\partial y_i)s$, and the equality follows from (6.4). Therefore we proved $E(R_i) > \rho$, a contradiction, so $y^* = 0$ which implies $y_0^* = w$.

Q.E.D.

We conclude this section by deriving an *explicit* solution for the optimal portfolio (y^*_0, y^*) in the case of only one risky asset, and with the risky return R being a two states random variable:

$$\text{Prob. }\{R = r_1\} = p, \quad \text{Prob. }\{R = r_2\} = 1-p \text{ with } r_1 > r_2,\ 0 < p < 1.$$

For this case, the portfolio problem for a OCE-maximizing investor is:

$$\max_{w \geq y \geq 0} \max_{\eta \in IR} \{-\rho y + \eta + pu(r_1 y - \eta) + (1-p)u(r_2 y - \eta)\}$$

$$:= \max_{0 \leq y \leq w} \max_{\eta \in IR} F(y, \eta)$$

The first order optimality conditions for an inner solution are:

$$\partial F/\partial y = -\rho + pr_1 u'(r_1 y - \eta) + (1-p) r_2 u'(r_2 y - \eta) = 0$$

$$\partial F/\partial \eta = 1 - pu'(r_1 y - \eta) - (1-p)u'(r_2 y - \eta) = 0$$

It is easy to verify that the solution of the above nonlinear system of equations is given by:

$$\bar{y} = \{\phi\ (\rho - r_2/p(r_1 - r_2)) - \phi\ (r_1 - \rho/(1-p)(r_1 - r_2))\}(r_1 - r_2)^4$$

---

4. This solution exists if and only if

$\rho - r_2/p(r_1 - r_2) \in$ range u' and $r_1 - \delta/(1-p)(r_1 - r_2) \in$ range u',

a necessary condition for which is $r_2 \leq \delta \leq r_1$. The condition is also sufficient if range $u' = [0, \infty)$ which is the case for the HARA utilities.

$$\bar{\eta} = r_1\bar{y} - \phi\,(\rho - r_2/\rho(r_1 - r_2))$$

where $\phi := (u')^{-1}$, the inverse function of u' which exists and is strictly decreasing by the strict concavity of u. Since $r_1 > r_2$, $\bar{y}$ is positive if and only if

$$\rho - r_2/p(r_1 - r_2) < r_1 - \rho/(1-p)(r_1 - r_2)$$

which can be rewritten as

$$\rho < pr_1 + (1-p)r_2 = E(R)$$

Hence the optimal investment in the risky asset is given explicitly by

$$y^* = \begin{cases} 0 \text{ if } \rho \geq E(R) \\ \bar{y} \text{ if } \rho < E(R) \text{ and } \bar{y} < w \\ w \text{ if } \rho < E(R) \text{ and } \bar{y} \geq w \end{cases}$$

7. The Case of Quadratic Utility

In this section we restrict our attention to the portfolio problems for the special family of quadratic utilities

$$u_\alpha(t) = t - \alpha t^2/2 \qquad (\alpha > 0), \quad \text{all } t < 1/\alpha \tag{7.1}$$

which are the α- modifications of $u(t) = t - t^2/2$.

Note that the restriction $t < 1/\alpha$ is necessary to guarantee that u_α is monotone increasing throughout its domain. The OCE corresponding to $u_\alpha(t)$ can be computed directly from (2.2),

$$S_\alpha(Z) = E(Z) - (1/2)\alpha \ \mathrm{Var}(Z) \tag{7.2}$$

for a random variable Z with support $[Z_{min}, Z_{max}]$ such that $Z_{max} \le 1/\alpha$.

The corresponding OCE functional

$$s_\alpha(y) := S_\alpha(R^T y)$$

of the random vector R is then

$$s_\alpha(y) = \mu^T y - (1/2)\alpha y^T V y \tag{7.3}$$

where μ is the expectation vector of R and V denotes its covariance matrix.

The portfolio problem is here the quadratic concave program:

$$(Q) \qquad \max_{y \ge 0} \{y^T(\mu - \rho e) - (\alpha/2) y^T V y\}$$

$$\sum_{i=1}^{n} y_i \le w$$

where $e = (1, \ldots, 1)^T$.

This simple formulation allows us to characterize a complete diversifier investor in terms of the problem's data without the assumption of independent risky assets. Indeed, a direct application of the optimality conditions for problem (Q) yields the following result.

Theorem 7.1 An investor with a quadratic utility $u_\alpha(t) = t - (\alpha t^2/2)$ is a complete diversifier, i.e. $y_1^*, \ldots y_n^* > 0$, $\Sigma y_i^* < w$, if and only if

$$V^{-1}(\mu - \rho e) > 0$$

$$(1/\alpha)e^T V^{-1}(\mu - \rho e) < w$$

Q.E.D.

We assume now that $\{R_i: i=1,\ldots,n\}$ are independent variables. Then for $u_\alpha(t) = t - \alpha t^2/2$, it can be easily checked that

$$S_\alpha(\sum_{i=1}^n R_i) = \sum_{i=1}^n S_\alpha(R_i) = \sum_{i=1}^n \mu_i - (1/2)\alpha\sigma_i^2$$

where $E(R_i) = \mu_i$ and Var $R_i = \sigma_i^2$. Therefore the corresponding OCE functional (7.3) is now

$$S_\alpha(y) = \sum_{i=1}^n \mu_i y_i - (1/2)\alpha\sigma_i^2 y_i^2$$

and hence the portfolio problem (Q) is given by the *separable* concave program:

$$\text{(SQ)} \qquad \max_{y \geq 0} \{\sum_{i=1}^n y_i(\mu_i - \rho) - (1/2)\alpha\sigma_i^2 y_i^2\}$$

$$\text{s.t.} \quad \sum_{i=1}^n y_i \leq w$$

In this case an explicit optimal solution is available.

Theorem 7.2 Assume $\{R_i\}$ are independent random variables, and let

$$u_\alpha(t) = t - \alpha t^2/2.$$

If

$$\sum_{i=1}^n \max\,[0, (\mu_i - \rho)/\alpha\sigma_i^2)] < w \tag{7.4}$$

then the optimal solution of (SQ) is

$$y_i^* = \max(0, (\mu_i - \rho)/\alpha\sigma_i^2) \qquad i = 1, \ldots, n \tag{7.5}$$

Corollary 7.1 $y_0^* > 0$ if and only if (7.4) holds.

Proof If (7.4) holds then $y_0^* = w - \Sigma y_i^* > 0$. Conversely we suppose that (7.4) does not hold and we show that $y_0^* = 0$.

Case 1: $\Sigma \max(0, \mu_i - \rho/\alpha\sigma_i^2) > w$. We claim then that $\delta^* = 0$ is impossible. If it was, from (7.6) the y_i^* are uniquely determined and are given by (7.5), but then $\Sigma y_i^* \leq w$ is violated. Hence $\delta^* > 0$ and from the complementary slackness condition it follows that $\Sigma y_i^* = w$ i.e. $y_0^* = 0$.

Case 2: $\Sigma \max(0, (\mu_i - \rho)/\alpha\sigma_i^2) = w$. Then $\delta^* = 0$ and y_i^* given in (7.5) satisfy the optimality conditions (7.6). But then $\Sigma y_i^* = w$, implying $y_0^* = 0$. Q.E.D.

All the results in this section remain valid if the assumption of the quadratic utility is replaced by the assumption of the so called *Hybrid model* i.e. u is the exponential utility and R is a normal random vector, since in that case the OCE coincides with (7.3), see Ben-Tal and Teboulle [4].

8. The Dual Portfolio Problem

The portfolio problem of the maximizing OCE investor with utility $u \in U$ has an interesting dual problem which is studied in this seciton.

Recall that the primal problem is the concave program

(P) $$\max_{\substack{y_0 \geq 0 \\ y \geq 0}} \{\rho y_0 + s(y)\}$$

subject to

$$y_0 + \sum_{i=1}^{n} y_i = w \tag{8.1}$$

$s(\cdot)$ being the OCE functional of the random vector R:

$$s(y) = S_u(R^T y)$$

We denote by $(R_i \bar{R}_i)$ the support of the random variable R_i.

Theorem 8.1 The dual problem of (P) is fiven by the one dimensional convex problem

(D) $\min \{\lambda w + P(\lambda) : \lambda \geq \max \{\rho, \max R_i\}\}$

where

$$P(\lambda) := \max_{y \geq 0} \{s(y) - \lambda \sum_{i=1}^{n} y_i\} \tag{8.2}$$

is a nonnegative nonincreasing convex function satsifying

(a) $P(\lambda) = 0$ if $\lambda \geq \max_i E(R_i)$

(b) $P(\lambda) > 0$ if $\lambda < \max_i E(R_i)$

Proof: The dual (D) is derived via Lagrangian duality. The Lagrangian for problem (P) is the function L: $\mathbf{R}_+^n \times \mathbf{R} \to \mathbf{R}$ with values:

$$L(y, \lambda) = \rho y_0 + s(y) + \lambda w - \lambda \Sigma y_i - \lambda y_0 \tag{8.3}$$

The dual objective function is defined by

$$h(\lambda) = \max_{\substack{y_0 \geq 0 \\ y \geq 0}} \{(\rho - \lambda) y_0 + s(y) - \lambda \Sigma y_i\} + \lambda w$$

$$= \lambda w + \max_{y_0 \geq 0} (\rho - \lambda) y_0 + \max_{y \geq 0} \{s(y) - \lambda \sum_{i=1}^{n} y_i\}$$

Then, the dual problem (D) is

$$\min_{\lambda \geq \rho} \{\lambda w + P(\lambda)\} \tag{8.4}$$

with $P(\lambda)$ defined in (8.2). Let $s_i(y_i) := s(0, \ldots, y_i, \ldots, 0)$ then

$$P(\lambda) \geq \max_{\substack{y_i \geq 0 \\ y_j = 0,\ j \neq i}} (s(y) - \lambda \Sigma y_i) = \max_{y_i \geq 0} (s_i(y_i) - \lambda y_i) \geq \max_{y_i \geq 0} (\underline{R}_i y_i - \lambda y_i) = \infty \text{ if } \underline{R}_i > \lambda.$$

The inequality following from $s_i(y_i) \geq y_i \underline{R}_i$ (see Proposition 2.1(c)). This explains the constraint $\lambda \geq \max_i \underline{R}_i$ in (D).

To prove (a) we note that

$$P(\lambda) = \max_{y \geq 0} \{s(y) - \lambda \Sigma y_i\} \geq s(0) = 0. \tag{8.5}$$

Also, by Proposition 2.1(b) $s(y) \leq \Sigma y_i E(R_i)$ with equality if and only if $y = 0$. This and (8.5) imply

$$0 \leq P(\lambda) \leq \max_{y \geq 0} \{ \sum_{i=1}^{n} y_i E(R_i) - \lambda \sum_{i=1}^{n} y_i\} = \sum_{i=1}^{n} \max_{y_i \geq 0} \{y_i (E(R_i) - \lambda)\} = 0 \quad \text{if}$$

$\lambda \geq E(R_i)$ for all i, proving (a).

To prove (b) recall from Proposition 2.2(b) that

$$s_i(0) = 0, \quad (\partial/\partial y_i) \quad s_i(0) = E(R_i)$$

hence by the assumption $\lambda < \max E(R_i)$ the function

$$v_i(y_i) := s_i(y_i) - \lambda y_i$$

satisfies

$$v_i(0) = 0, \; (d/dy_i)v_i(0) = E(R_i) - \lambda > 0$$

proving (b).

Finally, by (8.2) $P(\lambda)$ is a pointwise supremum of non increasing affine functions and hence is convex and non increasing.

Q.E.D.

Intrepretation of the Dual:

Consider a bank offering a saving plan in which λ dollars is paid at the end of the period per every dollar deposited. Thus if the investor's initial wealth w is deposited then his end of period wealth will be λw. To be competitive with the investment opportunities of the individual, the rate λ must be at least as large as the safe return per dollar ρ i.e.

$$\lambda \geq \rho$$

Moreover, to be competitive to the investment opportunity in each of the risky asset, whose return is R_i, λ must be at least as large as the worst realization of R_i i.e.

$$\lambda \geq \underline{R}_i \qquad \text{for all } i = 1, \ldots, n.$$

The quantity $s(y)$ is the value assigned by an individual having utility u to the amounts $y_1, \ldots, y_n$ invested in the risky assets. On the other hand the total portfolio value $\Sigma\, y_i$ when deposited in the bank is $\lambda \Sigma y_i$, thus the investor maximal *loss* by doing the latter is

$$\max_{y \geq 0} \; \{s(y) - \lambda \sum_{i=1}^{n} y_i\}$$

In order for the saving plan to be attractive to the individual the bank must add to λw the *extra bonus* which will offset the above potential loss. The objective of the bank is to minimize the total end of period payment to the investor, while being at the same time competitive to the investment opportunities in the capital market in the above sense. This is exactly modelled by the dual problem (D).

Notice that according to Theorem 8.1, the extra bonus is paid (i.e. $P(x) > 0$) if and only if the performance of the bank saving plan is worse than the mean peformance of one or more of the risky investment opportunities, i.e. $\lambda < E(R_i)$ for some i.

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The portfolio selection problem with one safe asset and n risky assets is analyzed via a new decision theoretic criterion: the Optimized Certainty Equivalent (OCE), recently introduced by the authors. Fundamental results in portfolio theory, previously studied under the Expected Utility criterion (EU), such as separation theorems, comparative statistics analysis, and threshold values for inclusion or exclusion of risky assets in the optimal portfolio, are obtained here. In contrast to the EU model, our results for the OCE maximizing investor do not impose restrictions on either the utility

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function or the underlying probability laws. We also derive a dual portfolio selection problem and provide it with a concrete economic interpretation.

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